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Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1. Fundamentals

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The well-known analogy between the Euler equations for steady flow of an inviscid incompressible fluid and the equations of magnetostatic equilibrium in a perfectly conducting fluid is exploited in a discussion of the existence and structure of solutions to both problems that have arbitrarily prescribed topology. A method of magnetic relaxation which conserves the magnetic-field topology is used to demonstrate the existence of magnetostatic equilibria in a domain $\mathscr{D}$ that are topologically accessible from a given field $\boldsymbol{B}_0(\boldsymbol{x})$ and hence the existence of analogous steady Euler flows. The magnetostatic equilibria generally contain tangential discontinuities (i.e. current sheets) distributed in some way in the domain, even although the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ may be infinitely differentiable, and particular attention is paid to the manner in which these current sheets can arise. The corresponding Euler flow contains vortex sheets which must be located on streamsurfaces in regions where such surfaces exist. The magnetostatic equilibria are in general stable, and the analogous Euler flows are (probably) in general unstable.

The structure of these unstable Euler flows (regarded as fixed points in the function space in which solutions of the unsteady Euler equations evolve) may have some bearing on the problem of the spatial structure of turbulent flow. It is shown that the Euler flow contains blobs of maximal helicity (positive or negative) which may be interpreted as 'coherent structures', separated by regular surfaces on which vortex sheets, the site of strong viscous dissipation, may be located.

1. Introduction

In a paper of great fundamental interest for fluid mechanics, Arnol'd (1974) has posed the following problem in magnetohydrodynamics,† which, by process of analogy, has an immediate bearing on the question of existence and structure of solutions of the steady Euler equations of fluid flow. Let $\mathscr{D}$ be a bounded domain in $\mathbb{R}^3$ (different possibilities are shown in figure 1) containing a viscous, but perfectly conducting, incompressible fluid. Suppose that, at time $t = 0$, the fluid is at rest and contains a magnetic field $\boldsymbol{B}_0(\boldsymbol{x})$ satisfying $\nabla \cdot \boldsymbol{B}_0 = 0$, $\boldsymbol{n} \cdot \boldsymbol{B}_0 = 0$ on $\partial\mathscr{D}$ (conditions that are then satisfied for all $t > 0$). In general, the Lorentz force $(\nabla \wedge \boldsymbol{B}_0) \wedge \boldsymbol{B}_0$ is rotational, and motion must ensue. The total energy (magnetic plus kinetic) must decrease owing to viscous dissipation, but the magnetic energy has a lower bound associated with any non-trivial topological structure in the field $\boldsymbol{B}_0(\boldsymbol{x})$ which is subsequently conserved since, by virtue of the perfect-conductivity assumption, $\boldsymbol{B}$-lines are frozen in the fluid. Hence the system must 'relax' to a state of magnetostatic equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ say, with $\boldsymbol{v} \equiv 0$, as $t \to \infty$, and evidently the

† Arnol'd attributes the conception of this problem to Ya. B. Zel'dovich.

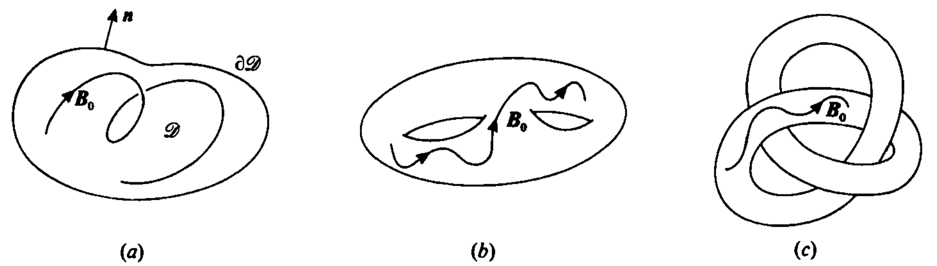


FIGURE 1. The magnetic-relaxation problem: the domain $\mathscr{D}$ may be (*a*) simply connected, (*b*) multiply connected, or even (*c*) multiply connected and of complex topology.

topology of $\boldsymbol{B}^{\mathrm{E}}$ is, in a sense that will be clarified in subsequent sections, the same as that of $\boldsymbol{B}_0$. A well-known analogy between the equations of magnetostatics, and the *steady* form of Euler's equations for inviscid flow, then implies the existence of an analogous 'Euler flow' $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ in $\mathscr{D}$, having the same topological structure as the arbitrary solenoidal field $\boldsymbol{B}_0$.

In earlier papers, Arnol'd (1965, 1966) had considered the general question of the topology of Euler flows $\boldsymbol{u}(\boldsymbol{x})$, and had shown that, under the assumption that $\boldsymbol{u}$ is an analytic function of $\boldsymbol{x}$, then either the streamlines (and vortex lines) lie on families of toroidal surfaces 'nearly everywhere' in $\mathscr{D}$, or $\boldsymbol{u}$ satisfies the equation curl $\boldsymbol{u} = \alpha \boldsymbol{u}$ for some constant α, throughout $\mathscr{D}$. This strong structural constraint seems incompatible with the more general type of flow conceived in the later paper, and Arnol'd (1974) concluded that the 'magnetic-relaxation' problem as described above would in general have no solution within the class of smooth vector fields.

In this paper we treat the magnetic relaxation problem *ab initio*, and we abandon the constraint that the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ should necessarily remain smooth. From a physical point of view, current sheets may of course survive in a perfect conductor (just as vortex sheets may survive in an inviscid fluid), and we shall show by particular example (in §5) just how such current sheets may arise even when the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is infinitely differentiable. We shall prove in §3 that the field *does* relax to an equilibrium field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$, which may in general contain such current sheets, distributed in some manner throughout $\mathscr{D}$. The field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ is, however, 'topologically accessible' from $\boldsymbol{B}_0(\boldsymbol{x})$ in a sense that will be made precise in §§2 and 3.

This result has some rather extraordinary consequences, particularly when interpreted in the context of steady Euler flows. It appears that for any given 'kinematically possible' flow $\boldsymbol{U}(\boldsymbol{x})$ in $\mathscr{D}$, with arbitrarily complex topology, there exists at least one Euler flow $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ with essentially the same streamline topology as $\boldsymbol{U}$. Since there is an uncountable infinity of possible flow topologies, this implies the existence of an equally uncountable infinity of topologically distinct solutions of the steady Euler equations for an arbitrary domain $\mathscr{D}$ (which may itself have arbitrarily complex topology).

It is interesting to note that it is the *streamline* topology, rather than the vortex-line topology, that can be freely prescribed. This runs counter to physical intuition based on the frozen-field character of the vorticity field in unsteady inviscid flow. However, the analogy on which this work is based is that between $\boldsymbol{B}$ and $\boldsymbol{u}$ in corresponding steady states, and does not extend to the evolution of perturbations about the steady states. This means that the stability properties of the Euler flows are not the same as those of the magnetostatic equilibria to which they correspond. An explicit example will be given in §4 to illustrate this important distinction. In general, however, the approach of this paper leaves open the question of stability of the Euler flows.

The results may have important consequences at a fundamental level in understanding the nature of turbulent flow. Insofar as viscosity may be neglected, steady solutions of the Euler equations may be regarded as the fixed points in the function space in which solutions of the unsteady Euler equations, which may have a turbulent character, evolve. Even if, as is probable, these fixed points are unstable, it is important to 'locate' them, and to understand the corresponding flow structure, and then to identify the 'heteroclinic orbits' in the function space connecting such points (the saddle connections), since these are known in general to be the seat of chaotic behaviour. The steady Euler flows that we shall find have a conceptually simple structure in which regions of maximum relative helicity (positive or negative) in which the streamlines are ergodic are separated by families of streamsurfaces which may include vortex sheets; these vortex sheets may be expected in general to be unstable, and, when viscosity is restored to the fluid, they are of course the seat of strong viscous dissipation. The regions of maximum relative helicity may then be recognized as fulfilling the role of 'coherent structures' in which the rate of energy dissipation is relatively low. Such ideas have been touched on previously (Moffatt 1984; Levich & Tsinober 1983), and the idea that coherent structures are helical in character receives observational support from the work of Tsinober & Levich (1983). This turbulence scenario, still somewhat speculative at this stage, is developed in §9 of this paper.

2. Constrained and unconstrained relaxation

The magnetic-relaxation problem outlined above is more subtle than it might appear at first sight because the topological constraints, although perhaps physically clear, are quite hard to characterize mathematically. It may therefore be helpful to consider first a different type of relaxation problem which illustrates in a clear and simple manner some of the features that are present in more disguised form in the magnetic context.

Consider an incompressible medium (which may for the moment be either an unstrained elastic dissipative solid or a viscous fluid) which at time $t = 0$ has non-uniform density $\rho_0(\boldsymbol{x})$, within the cylindrical domain

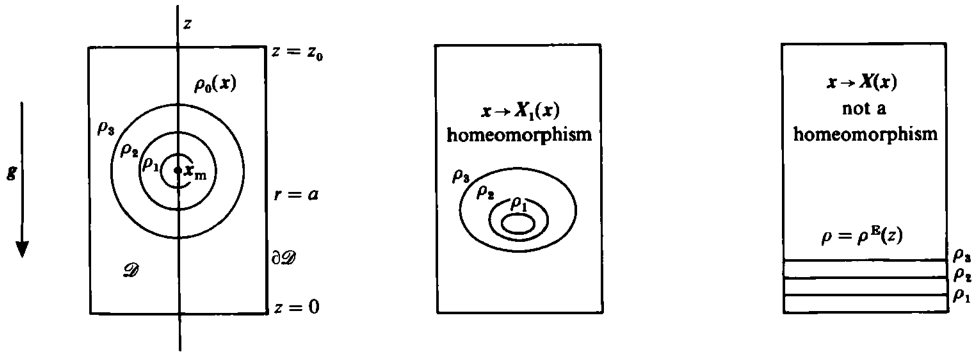
$$\mathscr{D}: 0 < z < z_0,\ 0 \leqslant r < a, \tag{2.1}$$

in a uniform gravitational field $\boldsymbol{g} = (0, 0, -g)$ (figure 2*a*). Since $\boldsymbol{g} \wedge \boldsymbol{\nabla}\rho_0 \neq 0$, this is obviously not an equilibrium configuration, and the medium will move with velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ (where $\boldsymbol{v}(\boldsymbol{x}, 0) = 0$); we suppose that $\boldsymbol{v} = 0$ on $\partial\mathscr{D}$. If $\rho(\boldsymbol{x}, t)$ is the density field then the equation of mass conservation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} \equiv \frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = 0, \tag{2.2}$$

i.e. the surfaces $\rho =$ const. are material surfaces: they are 'frozen' in the medium.

Suppose first that the medium is an elastic solid, unstrained at time $t = 0$, which dissipates energy when it moves within $\mathscr{D}$. To be specific, let us suppose that $\rho_0(\boldsymbol{x})$ is uniform except for a region of increased density near the centre of $\mathscr{D}$ as illustrated in figure 2(*a*). Then it is clear what must happen: the heavier part of the medium 'sags' downwards, causing a (net) displacement of the lighter part upwards. As $t \to \infty$ (and possibly after some oscillations) the medium settles down to an equilibrium in



(a) $t = 0$, $\rho_1 > \rho_2 > \rho_3$ (b) $t = \infty$, elastic solid (c) $t = \infty$, incompressible fluid

FIGURE 2. Gravitational relaxation of an incompressible medium with an initial density perturbation.

which the sum of the gravitational potential energy and the elastic strain energy is minimized (figure 2*b*). We may characterize this equilibrium by a mapping

$$x \to X_1(x), \tag{2.3}$$

where $X_1(x)$ is the final position of the particle initially at x. Since the elastic energy is finite, it is clear that this mapping must be continuous, and indeed uniformly continuous, i.e. for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|X_1(x) - X_1(y)| < \epsilon \quad \text{whenever } |x - y| < \delta, \tag{2.4}$$

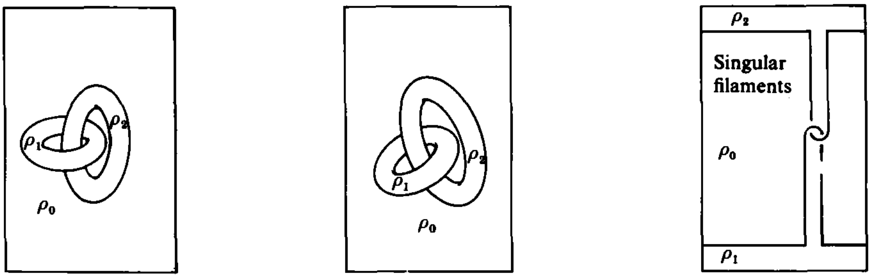
where x and y are any two points in $\bar{\mathscr{D}} (= \mathscr{D} \cup \partial\mathscr{D})$. It is moreover clear that the mapping (2.3) is one–one, that its inverse exists, and that this inverse is also uniformly continuous. In the language of topology, the mapping (2.3) is by virtue of these properties, a *homeomorphism*, which establishes a *topological equivalence* between the initial density field $\rho_0(x)$, and the final equilibrium density field $\rho^E(x)$, given by

$$\rho^E(X_1) = \rho_0(x). \tag{2.5}$$

This topological equivalence corresponds to the self-evident fact that the surfaces $\rho^E(x) = \text{const.}$ can be obtained from the surfaces $\rho_0(x) = \text{const.}$ by the continuous distortion of the medium $x \to X_1(x)$. The same would be true for 'more interesting' initial conditions, e.g. that indicated in figure 3(*a*), in which $\rho_0(x)$ is uniform except in two linked toroids, in one of which there is a density deficit, and in the other a density excess. This density field will again relax (via a homeomorphism) to an equilibrium field $\rho^E(x)$ that is topologically equivalent in a strict mathematical sense to $\rho_0(x)$.

Suppose now that the medium in $\mathscr{D}$ is a viscous fluid (still incompressible). Then there is no build-up of elastic energy, i.e. the relaxation process is unconstrained (except for the innocuous constraint of incompressibility). Relaxation therefore proceeds in an unimpeded manner until all the heavier fluid accumulates at the bottom of the cylinder; indeed the ultimate equilibrium as $t \to \infty$ (again possibly after some damped oscillations) is given by

$$\rho^E = \rho^E(z), \quad \frac{d\rho^E}{dz} \leqslant 0, \tag{2.6}$$



(*a*) $t = 0$, $\rho_1 > \rho_0 > \rho_2$ (*b*) $t = \infty$, elastic solid (*c*) $t \to \infty$, incompressible fluid

FIGURE 3. Same as figure 2, but with the initial density perturbations in linked toroids.

this being the configuration of minimum potential energy. The surfaces ρ^{E} = const. are now horizontal planes intersecting $\partial\mathscr{D}$ (figure 2*c*). These surfaces do not appear to be topologically equivalent to the surfaces $\rho_0(\boldsymbol{x})$ = const.! If we allow closure of the surfaces ρ^{E} = const. on the boundary $\partial\mathscr{D}$ then this topological equivalence survives, but in a restricted sense, since evidently material surfaces that were quite separate at $t = 0$ have now (in the limit $t = \infty$) been squashed together on $\partial\mathscr{D}$.

It is easy to see that this behaviour is related to non-continuity of the mapping function $\boldsymbol{X}(\boldsymbol{x})$ that takes the fluid particle initially at $\boldsymbol{x}$ to its final position $\boldsymbol{X}$. For example, let $\boldsymbol{x}_{\mathrm{m}}$ be the point of maximum density in figure 2(*a*); then any initial small sphere of fluid $|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{m}}| < \delta$ is ultimately spread over a layer on the bottom of the cylinder; hence the continuity condition (2.4) cannot be satisfied for $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{m}}$; i.e. the mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ is *not* a homeomorphism. Actually the mapping is discontinuous for all points $\boldsymbol{x}$ vertically below $\boldsymbol{x}_{\mathrm{m}}$, if the initial configuration of figure 2(*a*) is axisymmetric.

The situation is somewhat worse for the initial condition of figure 3(*a*); here the lighter fluid all ends up at the top of the cylinder, and the heavier fluid all ends up at the bottom. The mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ is then apparently discontinuous for all points $\boldsymbol{x}$ on two surfaces emanating from the curves of maximum and minimum density; and also for all points $\boldsymbol{x}$ on two sections of the toroids which are stretched without limit in the filaments of vanishing cross-section which are the final vestige of the initial linkage (figure 3*c*).†

A certain property of the final approach to equilibrium in these fluid situations is worth noting. The final stage in the transition from figure 2(*b*) to figure 2(*c*) involves the squeezing out of a thin film of lighter fluid by the descending heavier fluid. This is analogous to the well-known 'squeeze-film' problem of lubrication theory (see e.g. Moffatt 1977, §3.6). The film thickness decreases asymptotically as an inverse power of time (typically $t^{-\frac{1}{2}}$, although the precise power may depend on the particular geometry envisaged). It is obviously during this stage that unbounded straining of certain fluid elements occurs. It might be thought that the final stage of approach to equilibrium might be captured in terms of a small perturbation (stability) analysis about the equilibrium state. This, however, is not the case. The small-perturbation equations admit exponentially damped solutions which do not cover the power-law behaviour described above. The 'perturbation' corresponding to the final stage of the

† We ignore non-Newtonian effects, which would in practice resolve such singularities.

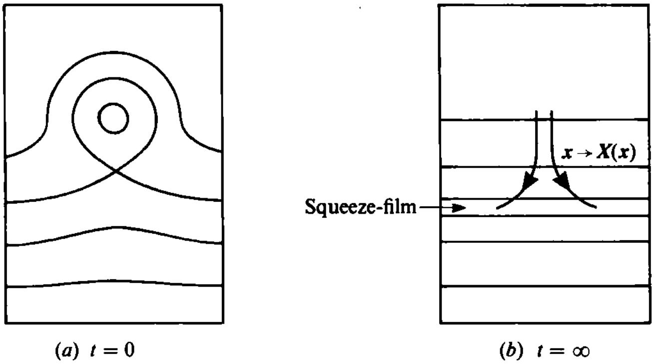


FIGURE 4. Same as figure 2, but with a density perturbation superposed on a stable density stratification; a squeeze-film forms in the interior of the fluid.

relaxation problem is in fact singular, and cannot be captured within the framework of a regular perturbation analysis.

Note that the squeeze-film may equally form in the interior of $\mathscr{D}$ if the initial conditions are different. For example, if $\rho_0(\boldsymbol{x})$ is stably stratified, but with a superposed density increment as indicated in figure 4(*a*), then the heavier blob will descend to the level at which it is in equilibrium with its surroundings (figure 4*b*), and a squeeze-film will form at about this level.

We shall find, in the context of the magnetic-relaxation problem, that the relaxation process is in certain respects intermediate between that of an elastic solid and that of an unconstrained fluid. The elastic behaviour is associated with the 'tension' in the magnetic lines of force ('$\boldsymbol{B}$-lines') which means that unbounded stretching of fluid elements in the direction of the convected magnetic field is not possible. On the other hand, fluid motion involving unbounded stretching of certain fluid elements in a direction perpendicular to the convected field is not excluded; the associated mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ relating initial and final states is then not continuous, and singular surfaces may form both on parts of $\partial\mathscr{D}$ and in the interior of $\mathscr{D}$. These singular surfaces are of course the current sheets referred to in §1.

Since the mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ is not in general a homeomorphism, the equilibrium field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ will not in general, within the conventional meaning of the term, be topologically equivalent to the initial field $\boldsymbol{B}_0(\boldsymbol{x})$. Nevertheless, the field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ is arrived at by the convective action of a velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ $(0 < t < \infty)$ which is certainly smooth (i.e. continuously differentiable) if the viscosity is sufficiently large, and which has finite total viscous energy dissipation. During this evolution, there is certainly no 'reconnection' of $\boldsymbol{B}$-lines, although for example two circular flux tubes, initially separate, may come into contact over a finite area in the final equilibrium state. We are therefore dealing with a somewhat weaker form of topological equivalence than that associated with homeomorphism. We need a new concept of *topological accessibility* rather than topological equivalence, a field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ being topologically accessible from a field $\boldsymbol{B}_0(\boldsymbol{x})$ if it can be obtained by the convective action on $\boldsymbol{B}_0(\boldsymbol{x})$ of a velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ satisfying the conditions of smoothness and finite dissipation specified above. We shall give further precision to this concept, which is of central importance to the argument, in §3.

3. The magnetic relaxation problem

Let us now consider the details of the magnetic relaxation problem described in §1. Here the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ evolve according to the equations of magnetohydrodynamics:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = -\nabla p+\boldsymbol{j}\wedge\boldsymbol{B}+\mu\nabla^2\boldsymbol{v}, \tag{3.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}\,(\boldsymbol{v}\wedge\boldsymbol{B}), \tag{3.2}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0, \tag{3.3}$$

where ρ is the fluid density (now assumed uniform), μ is the viscosity (which we may for convenience assume sufficiently large for the Reynolds number associated with the flow to be permanently of order unity or less), $p(\boldsymbol{x},t)$ is the pressure field, and

$$\boldsymbol{j} = \operatorname{curl}\boldsymbol{B} \tag{3.4}$$

is the current field (the conventional factor μ_0 being omitted for simplicity of notation). Equation (3.2) guarantees that $\boldsymbol{B}$-lines are frozen in the fluid, so that all knots and links in $\boldsymbol{B}$-lines are permanently conserved. The initial conditions are

$$\boldsymbol{B}(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x}), \quad \boldsymbol{v}(\boldsymbol{x},0) = 0, \tag{3.5}$$

where $\boldsymbol{B}_0(\boldsymbol{x})$ is an arbitrary smooth solenoidal field satisfying $\boldsymbol{n}\cdot\boldsymbol{B}_0 = 0$ on $\partial\mathscr{D}$. The boundary condition associated with (3.1) is

$$\boldsymbol{v} = 0 \quad \text{on } \partial\mathscr{D}, \tag{3.6}$$

and (3.2) then guarantees that

$$\boldsymbol{n}\cdot\boldsymbol{B} = 0 \quad \text{on } \partial\mathscr{D} \quad \text{for all } t \geqslant 0. \tag{3.7}$$

The problem thus defined may appear somewhat artificial in that, in most conducting fluids or plasmas, the magnetic resistivity η is of the same order of magnitude, if not much greater than, the kinematic viscosity $\nu = \mu/\rho$, and it may therefore seem unrealistic to take account of viscosity while neglecting resistivity. It is a situation where the end justifies the means: it will become apparent that the introduction of viscous dissipation is merely a mathematical expedient which is helpful in forcing the system towards a particular type of equilibrium, and thereby proving the existence of this equilibrium. Other types of dissipation might do equally well – e.g. if $\mu\nabla^2\boldsymbol{v}$ in (3.1) is replaced by $-k\boldsymbol{v}$, an analogous relaxation theory may be developed; the crucial feature of the argument, however, is that magnetic resistivity *is* neglected during the relaxation process. To be sure, once the magnetostatic equilibrium (with or without current sheets) is identified, the effects of non-zero resistivity may then be considered, and no doubt resistive instabilities of various kinds (Furth, Killeen & Rosenbluth 1963) will then lead to changes in the magnetic topology; but that is a quite separate issue, outside the scope of the present analysis.

The energy equation may be obtained by taking the scalar product of (3.1) with $\boldsymbol{v}$, (3.2) with $\boldsymbol{B}$, adding and integrating over $\mathscr{D}$. Defining the magnetic energy

$$M(t) = \frac{1}{2}\int_{\mathscr{D}} \boldsymbol{B}^2\,\mathrm{d}V, \tag{3.8}$$

the kinetic energy

$$K(t) = \frac{1}{2}\int_{\mathscr{D}} \rho v^2 \,\mathrm{d}V, \tag{3.9}$$

and the rate of viscous dissipation

$$\Phi(t) = \int_{\mathscr{D}} \mu(\operatorname{curl} \boldsymbol{v})^2 \,\mathrm{d}V, \tag{3.10}$$

this equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(t)+K(t)) = -\Phi(t), \tag{3.11}$$

so that the total energy certainly decreases monotonically for so long as $\boldsymbol{v} \not\equiv 0$.

Now the decrease of magnetic energy proceeds by a process of contraction of $\boldsymbol{B}$-lines (just the opposite of the dynamo process in which *increase* of magnetic energy is associated with *stretching* of $\boldsymbol{B}$-lines). Let us consider the details of this process for the case of a circular magnetic flux tube of radius a and initially small uniform cross-section ΔS, carrying flux Φ. With $B = \Phi/\Delta S$ in the tube, the magnetic energy is

$$\frac{1}{2}\int_{\text{tube}} \boldsymbol{B}^2 \,\mathrm{d}V = 2\pi^2 a^2 \Phi^2 / V, \tag{3.12}$$

where $V = 2\pi a\,\Delta S$ is the volume of the tube. If the tube contracts, keeping Φ and V constant, this energy obviously decreases, and this can be thought of as due to the release of the magnetic tension in the $\boldsymbol{B}$-lines. Note that if two flux tubes of equal volume V and of radii a, b ($b < a$) carrying fluxes Φ_1, Φ_2 ($\Phi_2 < \Phi_1$) are interchanged (figure 5) then there is a net reduction of magnetic energy

$$\frac{2\pi^2}{V}(a^2-b^2)(\Phi_1^2-\Phi_2^2). \tag{3.13}$$

This is the well-known interchange instability.

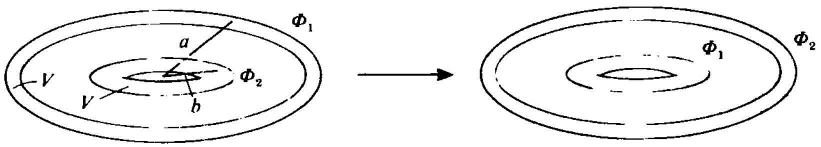


FIGURE 5. Interchange of two flux tubes, leading to a decrease of magnetic energy if $\Phi_1 > \Phi_2$.

Obviously therefore, $\boldsymbol{B}$-lines will continue to contract and thus to release magnetic energy for so long as an unimpeded 'contraction path' is available. If, however, the topology of the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is non-trivial, in the sense that not all of the $\boldsymbol{B}$-lines can shrink to a point without cutting other $\boldsymbol{B}$-lines (see e.g. figure 9a on p. 371), then $M(t)$ cannot decrease below some bound $M_{\min}(>0)$ determined by this topology. One possible measure of the 'degree of linkage' of $\boldsymbol{B}$-lines (Moffatt 1969) is given by the helicity of the field

$$\mathscr{H} = \int_{\mathscr{D}} \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V, \tag{3.14}$$

where $\boldsymbol{A}$ is a vector potential for $\boldsymbol{B}$ satisfying

$$\nabla \wedge \boldsymbol{A} = \boldsymbol{B}, \quad \nabla\cdot\boldsymbol{A} = 0. \tag{3.15}$$

This helicity is constant under frozen-field evolution (Woltjer 1958), and it is in general non-zero for fields of non-trivial topology. Arnol'd (1974) (see also Moffatt

1969, §4) has identified $\mathscr{H}$ with a generalized 'asymptotic' form of the Hopf invariant and has indicated how the invariance of $\mathscr{H}$ places a lower bound on $M(t)$. First, by the Schwartz inequality,

$$\int_{\mathscr{D}} \boldsymbol{A}^2 \,\mathrm{d}V \int_{\mathscr{D}} \boldsymbol{B}^2 \,\mathrm{d}V \geqslant \left(\int_{\mathscr{D}} \boldsymbol{A}\cdot\boldsymbol{B} \,\mathrm{d}V\right)^2 = \mathscr{H}^2. \tag{3.16}$$

Secondly, by standard methods of the calculus of variations (for a detailed treatment of an analogous problem see Roberts 1967, chap. 3), we may show that

$$\int_{\mathscr{D}} \boldsymbol{B}^2 \,\mathrm{d}V \geqslant q_0^2 \int_{\mathscr{D}} \boldsymbol{A}^2 \,\mathrm{d}V, \tag{3.17}$$

where $q_0^2 (> 0)$ is the smallest eigenvalue of the problem

$$\left.\begin{aligned} (\nabla^2 + q^2)\boldsymbol{A} &= 0 \quad \text{in } \mathscr{D},\\ \nabla \wedge \boldsymbol{A} &= 0 \quad \text{outside } \mathscr{D},\\ [\boldsymbol{A}]^+_- &= 0 \quad \text{across } \partial\mathscr{D},\\ \boldsymbol{A} &\rightarrow 0 \quad \text{as } |\boldsymbol{x}| \rightarrow \infty. \end{aligned}\right\} \tag{3.18}$$

Hence, combining (3.16) and (3.17), we have

$$M(t) \geqslant \tfrac{1}{2} q_0 |\mathscr{H}|, \tag{3.19}$$

which provides the required lower bound.

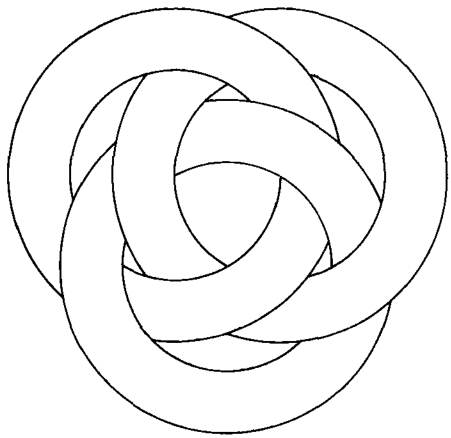


FIGURE 6. The Borromean ring configuration; for such configurations involving 'higher-order linkage', all helicity invariants vanish, but there is nevertheless a topological constraint on magnetic energy decay.

Note that for an arbitrary $\boldsymbol{B}_0(\boldsymbol{x})$ the *actual* lower bound for $M(t)$ may be very much greater than that given by (3.19). For example, as recognized by Arnol'd (1974), even if $\mathscr{H} = 0$, there may exist subdomains $\mathscr{D}_i(t)$ of $\mathscr{D}$ such that $\partial\mathscr{D}_i(t)$ is a magnetic surface, so that the helicity $\mathscr{H}_i$ associated with the subdomain is invariant (Moffatt 1969). This will provide a lower bound analogous to (3.19). More subtly, there are configurations, such as the Borromean ring configuration (see figure 6) for which $\mathscr{H}_i = 0$ for every subdomain $\mathscr{D}_i(t)$ within a magnetic surface, and which nevertheless exhibit a higher-order linkage than that which contributes to helicity integrals. There is undoubtedly a lower bound for $M(t)$ in such situations, but it cannot be expressed in a form as simple as (3.19).

The important point, however, is that, if the topology is non-trivial, then a lower bound $M_{\min}$ for $M(t)$ (and *a fortiori* for $M(t)+K(t)$) certainly exists. Hence since $M(t)+K(t)$ is a monotonic decreasing function, bounded below, it must tend to a constant, $M^{\mathrm{E}} \geqslant M_{\min}$, as $t \to \infty$; in this asymptotic situation the dissipation must vanish, and so

$$\boldsymbol{v} \equiv 0 \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) \not\equiv 0, \tag{3.20}$$

and, from (3.1), it is evident that $\boldsymbol{B}^{\mathrm{E}}$ is a magnetostatic field satisfying

$$\boldsymbol{j}^{\mathrm{E}} \wedge \boldsymbol{B}^{\mathrm{E}} = \nabla p^{\mathrm{E}}, \tag{3.21}$$

with curl $\boldsymbol{B}^{\mathrm{E}} = \boldsymbol{j}^{\mathrm{E}}$, for some $p^{\mathrm{E}}(\boldsymbol{x})$. We emphasize that in this equilibrium state the field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ may have surfaces of tangential discontinuity imbedded in $\mathscr{D}$. If Σ is such a surface, however, then $\boldsymbol{n}\cdot\boldsymbol{B}^{\mathrm{E}} = 0$ on both sides of Σ (since otherwise there would be an infinite Lorentz force component tangential to the sheet), and the jump condition related to (3.21) is satisfied, *viz*

$$[p^{\mathrm{E}} + \tfrac{1}{2}\boldsymbol{B}^{\mathrm{E}}\cdot\boldsymbol{B}^{\mathrm{E}}]_-^+ = 0 \quad \text{across } \Sigma. \tag{3.22}$$

The precise manner in which such current sheets may arise, even if the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is infinitely differentiable, will be considered in §5. For the moment, we simply observe that the appearance of such singularities must evidently be associated with non-continuity of the mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ associated with the net displacement of the fluid during the relaxation process.

In terms of this mapping, the final field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ is related to the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ by

$$B_i^{\mathrm{E}}(\boldsymbol{X}) = B_{0j}(\boldsymbol{x})\frac{\partial X_i}{\partial x_j}, \tag{3.23}$$

the Lagrangian counterpart of the frozen-field equation (3.2). Now, although $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{X})$ may have discontinuities, it is physically obvious that $|\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{X})|$ is bounded in $\mathscr{D}$ (a singularity of $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{X})$ would require infinite stretching of $\boldsymbol{B}$-lines, which appears to be incompatible with the nature of the relaxation process); hence, from (3.23), $\boldsymbol{X}(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}$ in the direction of $\boldsymbol{B}_0(\boldsymbol{x})$. However, $\boldsymbol{X}(\boldsymbol{x})$ need not be differentiable in directions perpendicular to $\boldsymbol{B}_0(\boldsymbol{x})$. Note, moreover, that

$$\frac{\partial B_i^{\mathrm{E}}}{\partial X_k} = \frac{\partial}{\partial x_m}\left(B_{0j}(\boldsymbol{x})\frac{\partial X_i}{\partial x_j}\right)\frac{\partial x_m}{\partial X_k}, \tag{3.24}$$

so that discontinuities of $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{X})$ may appear if the inverse mapping $\boldsymbol{X} \to \boldsymbol{x}(\boldsymbol{X})$ is non-differentiable.

The field $\boldsymbol{B}(\boldsymbol{x}, t)$ for any *finite* $t > 0$ can be pictured more simply. The particle displacement $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x}, t)$ for finite t *is* a homeomorphism, and the corresponding $\boldsymbol{B}(\boldsymbol{x}, t)$ is topologically equivalent in a strict sense to $\boldsymbol{B}_0(\boldsymbol{x})$. As $t \to \infty$, however, just as in the simpler example of §2, the limit mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ may become non-continuous, and the limit field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ is then not strictly topologically equivalent to $\boldsymbol{B}_0(\boldsymbol{x})$. It is nevertheless *topologically accessible* from $\boldsymbol{B}_0(\boldsymbol{x})$ in the sense defined in §2: it is obtained by distortion of the field $\boldsymbol{B}_0(\boldsymbol{x})$ by a smooth solenoidal velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ $(0 < t < \infty)$, whose dissipation function $\Phi(t)$, given by (3.10), satisfies

$$\int_0^\infty \Phi(t)\,\mathrm{d}t < \infty. \tag{3.25}$$

Under these conditions all links and knots in $\boldsymbol{B}_0(\boldsymbol{x})$ are faithfully carried over to $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$, and we may say (loosely) that $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ *has the same topology* as $\boldsymbol{B}_0(\boldsymbol{x})$.

It is clear that in general this magnetostatic equilibrium will be stable to small

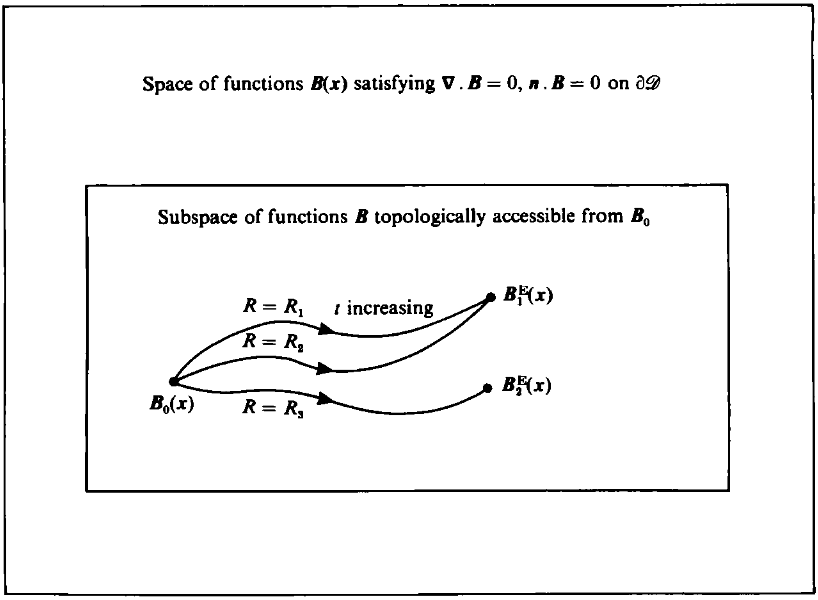


FIGURE 7. Schematic indication of the approach to a magnetostatic equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ in the subspace of fields topologically accessible from the initial field $\boldsymbol{B}_0(\boldsymbol{x})$. The equilibrium field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ need not be unique, and different relaxation procedures may lead to different stable equilibria.

perturbations governed by (3.1)–(3.5) linearized about the equilibrium state. We should note that the equilibrium field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ of given topology may not be unique, and in fact if the relaxation process is carried out for different values of the dimensionless number

$$R = \overline{B}_0 L\rho/\mu, \tag{3.26}$$

where $\overline{B}_0$ is the r.m.s. value of $\boldsymbol{B}_0$ in $\mathcal{D}$ and L is a lengthscale characterizing $\mathcal{D}$, then different equilibria may be located. If we change R then in effect we change the path in function space (figure 7) leading to magnetostatic equilibrium; but in all cases we remain in the subspace of vector fields that are topologically equivalent to or accessible from $\boldsymbol{B}_0$, and, in all cases, decay to *some* stable magnetostatic equilibrium is inevitable. Some particular examples of this process are considered in detail in §§4 and 5.

4. The case of trivial topology

Suppose that $\mathcal{D}$ is simply connected and that the lines of forces of $\boldsymbol{B}_0(\boldsymbol{x})$ are all unknotted closed curves which may each be shrunk to a point without cutting any other line of force. This is the case of 'trivial topology' for which no topological constraint impedes the energy decay process. We therefore anticipate that $M(t)+E(t) \to 0$ as $t \to \infty$; it is instructive to examine precisely how this decay occurs.

To fix ideas, suppose that $\mathcal{D}$ is the cylinder

$$r < a, \quad 0 < z < z_0, \tag{4.1}$$

of volume $V_0 = \pi a^2 z_0$, in cylindrical polar coordinates (r, φ, z), and that

$$\boldsymbol{B}_0(\boldsymbol{x}) = (0, B_0(r,z), 0), \tag{4.2}$$

where $B_0(r,z)$ has bounded support $V_B < V_0$, i.e. $\boldsymbol{B}_0$ consists of a single circular flux tube within $\mathcal{D}$ (figure 8*a*). Arnol'd (1974) briefly discusses this problem (with acknowledgement to Ya. B. Zel'dovich).

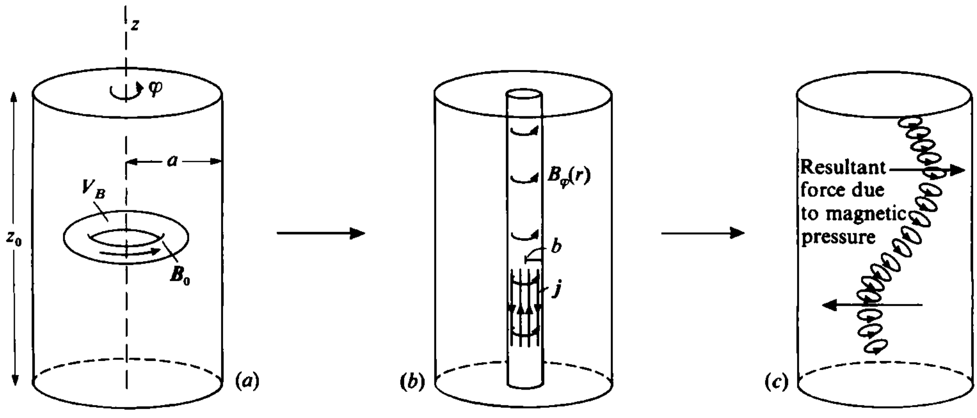


FIGURE 8. Magnetic relaxation for the trivial topology: (*a*) the initial state; (*b*) the axisymmetric state of minimum energy; (*c*) non-axisymmetric instability.

The magnetic energy associated with the field (4.2) is

$$M_0 = \pi \int_{S_B} \left(\frac{B_0}{r}\right)^2 r^3 \,\mathrm{d}r\,\mathrm{d}z, \tag{4.3}$$

where S_B is the cross-section of the torus V_B. We write it in this way because, in the subsequent frozen-field evolution, $B_\varphi(r, z)/r$ remains constant on any circle $r = r(t)$ moving with the fluid. Because of the weighting factor r^3 in (4.3) the energy can therefore decrease through shrinkage of each and every circular line of force. For so long as the motion remains axisymmetric the minimum is achieved when V_B is deformed to cylindrical shape, the field lines having rearranged themselves so that B_φ is a function of r only and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi(r)}{r}\right) \leqslant 0. \tag{4.4}$$

The magnetic energy in this new configuration is then

$$M_1 = \pi z_0 \int_0^b \left(\frac{B_\varphi}{r}\right)^2 r^3 \,\mathrm{d}r, \tag{4.5}$$

where $\pi b^2 z_0 = V_B$. The associated current is parallel to Oz, the total current in the tube being zero (figure 8*b*).

This magnetostatic equilibrium is, however, clearly unstable to non-axisymmetric disturbances since the magnetic pressure acts as indicated in figure 8(*c*) in such a way as to increase the perturbation. The increase in the length L of the tube is compensated by a decrease in cross-section. It is sufficient to consider the special case in which

$$\frac{B_\varphi}{r} = \begin{cases} \text{const.} & (r \leqslant b), \\ 0 & (r > b), \end{cases} \tag{4.6}$$

where $b \ll z_0$, i.e. the current tube is long and thin. In this case, any instability of the tube of wavelength $\lambda \gg b$ will increase its length to $L(> z_0)$ and decrease its cross-sectional radius to δ, where, by conservation of volume,

$$L\delta^2 \approx z_0 b^2. \tag{4.7}$$

During this distortion the field distribution (4.6) is maintained, provided r now represents distance from the deformed axis of the tube. Hence to leading order in the small parameter b/λ, the magnetic energy in the perturbed state is given by

$$M \approx \pi L \int_0^\delta \left(\frac{B_\varphi}{r}\right)^2 r^3 \,\mathrm{d}r \approx \tfrac{1}{4} L \delta^4 \left(\frac{B_\varphi}{r}\right)^2 \approx M_1 \frac{z_0}{L}. \tag{4.8}$$

Hence M decreases as L increases, and the tube will continue to lengthen. Since instabilities can now continue in a similar way on any wavelength large compared with the ever-decreasing cross-sectional radius δ, M can clearly continue to decrease, the limiting value being $M_{\min} = 0$ (when $L = \infty$). Note that, although $\boldsymbol{B} \to 0$ in $\mathscr{D}$ for all $\boldsymbol{x}$, the current field has an interesting structure in the limit $t \to \infty$, being concentrated in a tube of infinite length ($L \to \infty$) and vanishing cross-sectional radius ($\delta \to 0$); since (4.7) continues to hold in the limit, the Hausdorff dimension of the set of points at which $\boldsymbol{j} \neq 0$ in the limit $t \to \infty$ remains equal to 3.

If the initial field of the form (4.2) fills the domain $\mathscr{D}$ (i.e. $V_B = V_0$) then, again by regarding the field as an assembly of flux tubes, it is easy to see that non-axisymmetric instabilities may in a similar way lead to a decrease of magnetic energy, and that there is no stable magnetostatic equilibrium other than the zero field, which is, surprisingly, topologically accessible from $\boldsymbol{B}_0(\boldsymbol{x})$.

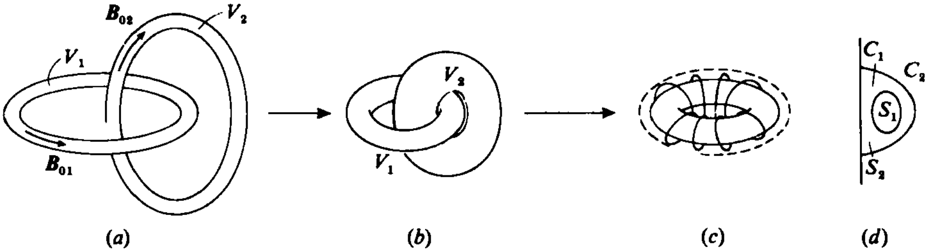


FIGURE 9. Magnetic relaxation for simple-linkage topology: (*a*) the initial state; (*b*) the stage at which the topological constraint impedes the relaxation process; (*c*) the axisymmetric state of minimum magnetic energy; (*d*) cross-section by a plane through the axis of symmetry.

5. The case of 'simple-linkage' topology

The next simplest case to consider (figure 9*a*) is that in which $\boldsymbol{B}_0(\boldsymbol{x})$ can be decomposed into two fields

$$\boldsymbol{B}_0(\boldsymbol{x}) = \boldsymbol{B}_{01}(\boldsymbol{x}) + \boldsymbol{B}_{02}(\boldsymbol{x}), \tag{5.1}$$

each of which separately has trivial topology, but with the toroidal support V_1 of $\boldsymbol{B}_{01}$ being simply linked with the toroidal support V_2 of $\boldsymbol{B}_{02}$. The field distribution over each torus cross-section may be chosen so that the field $\boldsymbol{B}_0(\boldsymbol{x})$ is smooth (and indeed, if required, infinitely differentiable). The helicity of this field distribution is

$$\mathscr{H} = 2\Phi_1 \Phi_2, \tag{5.2}$$

where Φ_1, Φ_2 are the total fluxes of $\boldsymbol{B}_{01}$, $\boldsymbol{B}_{02}$ around their respective tori (Moffatt 1969), and so (3.19) provides an immediate lower bound for the magnetic energy.

Again, as in §4, the field lines tend to shrink, the volumes V_1 and V_2 being conserved. Obviously, this shrinkage is restricted if and when the two rings 'make contact' (figure 9*b*). Further decrease of the magnetic energy is still possible, however, through redistribution of the $\boldsymbol{B}$-lines in V_2 (say) more uniformly close to the surface of V_1; the minimum is attained in the axisymmetric configuration of figure 9(*c*). The final stage of approach to this equilibrium is a squeeze-film flow. Note here that an alternative equilibrium is given by interchanging the roles of V_1 and V_2, an indication of the non-uniqueness of topologically accessible equilibria.

A section by a plane through the axis of symmetry of the equilibrium situation is shown in figure 9(*d*): S_1 (with boundary C_1) is the section of V_1, and S_2 (bounded internally by C_1 and externally by C_2) is the section of V_2. The field in V_1 in the minimizing state has the form

$$\boldsymbol{B}_1^{\mathrm{E}} = (0, B_\varphi(r), 0), \tag{5.3}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi(r)}{r}\right) \leqslant 0 \tag{5.4}$$

for reasons given in §4, and the field in V_2 has the form

$$\boldsymbol{B}_2^{\mathrm{E}} = \boldsymbol{\nabla} \wedge (0, A_\varphi(r,z), 0), \tag{5.5}$$

the lines of force in V_2 being given by

$$rA_\varphi(r,z) = \text{const.} \tag{5.6}$$

The shape of $\partial\mathscr{D}$ is obviously irrelevant for this equilibrium state. The magnetic energy of the configuration is

$$M = \pi \int_{S_1} \left(\frac{B_\varphi}{r}\right)^2 r^3 \,\mathrm{d}r\,\mathrm{d}z + \pi \int_{S_2} \frac{1}{r} (\boldsymbol{\nabla}(rA_\varphi))^2 \,\mathrm{d}r\,\mathrm{d}z, \tag{5.7}$$

and minimization of this expression involves a compromise between contraction of the $\boldsymbol{B}$-lines in V_2 (which tends to make C_1 circular) and contraction of the $\boldsymbol{B}$-lines in V_1 (which tends to make C_1 more 'D-shaped').

Note here that B_φ/r is certainly non-zero on the part of C_1 nearest to the axis Oz; moreover $\boldsymbol{B}_2^{\mathrm{E}}$ is also certainly non-zero approaching C_1 from S_2. Hence both meridional and azimuthal components of $\boldsymbol{B}^{\mathrm{E}}$ are discontinuous across C_1, and there is therefore a helical current sheet on the surface of V_1; this is true even if $\boldsymbol{B}_0(\boldsymbol{x})$ is infinitely differentiable, and this is a sure indication that the mapping $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x})$ relating initial and final configurations is in this case not a homeomorphism.

6. The role of helicity invariants

The important role of Woltjer's (1958) helicity invariant in impeding the relaxation process has already been noted in §2. It was in fact shown by Woltjer that minimization of M subject *only* to the constraint of invariant total helicity leads to a field satisfying

$$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \alpha \boldsymbol{B}, \quad \alpha = \text{const.} \tag{6.1}$$

Such a field is generally not topologically accessible from an arbitrary initial field, since the minimization process may have failed to conserve the helicities $\mathscr{H}_i$ associated with sub-domains $\mathscr{D}_i$ inside magnetic surfaces (see the discussion of §2). If these more detailed constraints are imposed (Taylor 1974), then minimization of M leads to a field satisfying

$$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \alpha(\boldsymbol{x})\,\boldsymbol{B} \quad \text{with } \boldsymbol{B}\cdot\boldsymbol{\nabla}\alpha = 0. \tag{6.2}$$

This field may still not be topologically accessible from $\boldsymbol{B}_0$ by a smooth volume-preserving flow, simply because the constraint of conservation of volume of flux tubes has not been incorporated. This additional constraint is built into (3.2) when $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, and minimization of M subject to the full constraint implied in (3.2) leads, as we have shown, to a magnetostatic field satisfying

$$\boldsymbol{j} \wedge \boldsymbol{B} = \boldsymbol{\nabla} p, \quad \nabla\cdot\boldsymbol{B} = 0. \tag{6.3}$$

7. The topology of chaos

As we have repeatedly observed, the topology of the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is quite arbitrary; its field lines may be closed curves; or they may cover surfaces of arbitrary topology embedded within $\mathscr{D}$; or, in some subdomains $\mathscr{D}_i$ of $\mathscr{D}$, the field lines may be ergodic, passing arbitrarily near each point of the subdomain if continued far enough. These are topological properties which are conserved during the relaxation process described by (3.1)–(3.5). The equilibrium field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ therefore has the same topology as $\boldsymbol{B}_0$.

Now from (3.21), $\boldsymbol{B}^{\mathrm{E}}\cdot\nabla p^{\mathrm{E}} = 0$, so that, provided $\nabla p^{\mathrm{E}} \neq 0$, $\boldsymbol{B}^{\mathrm{E}}$ lines lie on surfaces $p^{\mathrm{E}} = \text{const.}$ If $\nabla p^{\mathrm{E}} = 0$ in some subdomain $\mathscr{D}'$ of $\mathscr{D}$ then $\boldsymbol{j}^{\mathrm{E}} = \alpha(\boldsymbol{x})\,\boldsymbol{B}^{\mathrm{E}}$ with $\boldsymbol{B}^{\mathrm{E}}\cdot\nabla\alpha = 0$ within this subdomain; so that, provided $\nabla\alpha \neq 0$, $\boldsymbol{B}^{\mathrm{E}}$ lines now lie on surfaces $\alpha = \text{const.}$ If $\nabla\alpha = 0$ in some subdomain $\mathscr{D}''$ of $\mathscr{D}'$ then $\boldsymbol{j}^{\mathrm{E}} = \alpha\boldsymbol{B}^{\mathrm{E}}$ with $\alpha = \text{const.}$ in $\mathscr{D}''$; only in this case may $\boldsymbol{B}^{\mathrm{E}}$ be ergodic in $\mathscr{D}''$ (or in some part of $\mathscr{D}''$).†

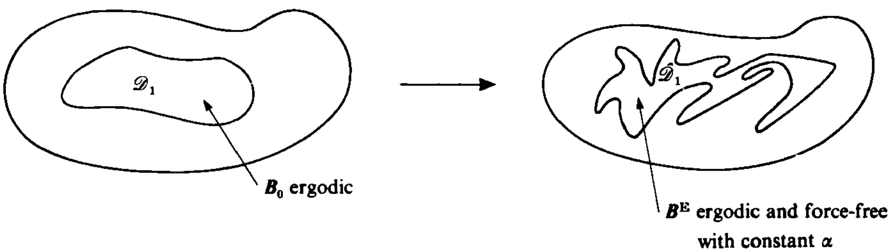


FIGURE 10. Schematic representation of the relation between the initial and final states when $\boldsymbol{B}_0(\boldsymbol{x})$ is ergodic in a subdomain $\mathscr{D}_1$ of $\mathscr{D}$; $\mathscr{D}_1$ maps to a subdomain $\mathscr{D}_1^{\mathrm{E}}$, in which $\boldsymbol{B}^{\mathrm{E}}$ is force-free with constant α.

We are therefore driven to the extraordinary conclusion that a field $\boldsymbol{B}_0(\boldsymbol{x})$ of arbitrarily complex topology can be continuously deformed to a magnetostatic equilibrium field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$, and that if $\boldsymbol{B}_0(\boldsymbol{x})$ is ergodic in any subdomain $\mathscr{D}_1$ that maps into a subdomain $\hat{\mathscr{D}}_1$ under the mapping $\boldsymbol{x}\to\boldsymbol{X}(\boldsymbol{x})$ then

$$\nabla \wedge \boldsymbol{B}^{\mathrm{E}} = \alpha\boldsymbol{B}^{\mathrm{E}} \quad \text{with } \alpha = \text{const. in } \hat{\mathscr{D}}_1. \tag{7.1}$$

The situation is shown schematically in figure 10.

As noted by Arnol'd (1974), the eigenfunctions of (7.1) for a *given* domain are few; however, the domain $\hat{\mathscr{D}}_1$ is not 'given', but its shape is determined by the whole relaxation process. We must conclude that, whatever the topology of $\boldsymbol{B}_0$ may be within the ergodic subdomain $\mathscr{D}_1$, this subdomain will so adjust itself during the relaxation process that the field does ultimately attain a form satisfying (7.1) throughout the image domain $\hat{\mathscr{D}}_1$. If the initial topology in $\mathscr{D}_1$ is complex, we may anticipate that the *geometry* of $\hat{\mathscr{D}}_1$ may be equally complex.

8. The implications for Euler flows

The Euler equations for inviscid flow of an incompressible fluid of unit density in a domain $\mathscr{D}$ are

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \wedge \boldsymbol{\omega} - \nabla h, \quad \nabla\cdot\boldsymbol{u} = 0, \tag{8.1}$$

where $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{u}$ and $h = p + \frac{1}{2}\boldsymbol{u}^2$. We assume, moreover, that $\boldsymbol{n}\cdot\boldsymbol{u} = 0$ on $\partial\mathscr{D}$. Any steady solution $\boldsymbol{u}(\boldsymbol{x})$ satisfies

$$\boldsymbol{u} \wedge \boldsymbol{\omega} = \nabla h. \tag{8.2}$$

† For an example of a space-periodic field $\boldsymbol{u}$ for which curl $\boldsymbol{u} = \alpha\boldsymbol{u}$ with $\alpha = \text{const.}$, and for which the streamlines are ergodic in a subdomain, see Hénon (1966) and Dombre *et al.* (1985).

Any particular steady solution $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ of these equations may be described as an Euler flow in $\mathscr{D}$.

Equations (8.2) exhibit an obvious analogy with the magnetostatic equations (6.3), the analogy being between the variables

$$\boldsymbol{B} \leftrightarrow \boldsymbol{u}, \quad \boldsymbol{j} \leftrightarrow \boldsymbol{\omega}, \quad p \leftrightarrow h_0 - h, \tag{8.3}$$

for some constant h_0. Hence to every magnetostatic equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$, there corresponds an Euler flow $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$, and *vice versa*.

We now formalize the result of §3 as a theorem in the context of Euler flows:

THEOREM. *Let $\boldsymbol{U}(\boldsymbol{x})$ be an arbitrary smooth solenoidal field in $\mathscr{D}$ satisfying $\boldsymbol{n}\cdot\boldsymbol{U} = 0$ on $\partial\mathscr{D}$. Then there exists at least one Euler flow $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ in $\mathscr{D}$ that is topologically accessible from $\boldsymbol{U}(\boldsymbol{x})$.*

This means that there exists a mapping $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x})$ that can be interpreted as the net displacement associated with a smooth velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ of finite total dissipation $(0 < t < \infty)$ such that

$$u_i^{\mathrm{E}}(\boldsymbol{x}) = U_j(\boldsymbol{x}) \frac{\partial X_i}{\partial x_j}. \tag{8.4}$$

Hence the streamlines of $\boldsymbol{U}$ map faithfully onto the streamlines of $\boldsymbol{u}^{\mathrm{E}}$. (Note that topological accessibility *does not* imply dynamical accessibility, which would require that the *vortex* lines of $\boldsymbol{U}$ be deformable to the *vortex* lines of $\boldsymbol{u}^{\mathrm{E}}$.)

As for the case of the magnetostatic equilibria, this theorem immediately implies the existence of an uncountable infinity of topologically distinct Euler flows. In particular, given any knot $\mathscr{K}$ in $\mathscr{D}$, there is an Euler flow that is topologically accessible from a flow $\boldsymbol{U}(\boldsymbol{x})$ along a closed streamtube centred on $\mathscr{K}$! Such a flow (like the magnetostatic equilibrium considered in §4) is unaffected by the boundary of the domain $\partial\mathscr{D}$, and can equally exist in an unbounded domain.

The question of the stability of Euler flows is a difficult one, which is in no way settled by appeal to the magnetostatic analogy. This is because perturbations about a steady state are governed by (8.1), which conserves vortex-line topology, in distinct conflict with the $\boldsymbol{B} \leftrightarrow \boldsymbol{u}$ analogy. This important distinction between Euler flows and the magnetostatic equilibria to which they correspond may be most readily appreciated by appeal to the particular example treated in §4. The Euler flow corresponding to the field of figure 8(*b*) is of the form

$$\boldsymbol{u}^{\mathrm{E}} = (0, u_\varphi(r), 0), \tag{8.5}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\varphi(r)}{r}\right) \leqslant 0. \tag{8.6}$$

This flow is unstable to axisymmetric perturbations if, by the Rayleigh criterion,

$$\frac{\mathrm{d}}{\mathrm{d}r}(r u_\varphi(r)) < 0. \tag{8.7}$$

Thus, for example, if

$$u_\varphi = \omega_0 r \,\mathrm{e}^{-r^2/b^2} \tag{8.8}$$

then (8.6) is satisfied for all r, and (8.7) is satisfied for $r > b$. Hence the flow is unstable to axisymmetric perturbations, although the corresponding magnetostatic equilibrium is stable to axisymmetric perturbations. (Conversely, this flow is probably *stable* to non-axisymmetric perturbations, although the corresponding magnetostatic equilibrium is *unstable* to such perturbations!)

9. Speculations concerning the structure of turbulence

As indicated in §1, insofar as viscous effects may be neglected, Euler flows may be regarded as fixed points in the function space in which unsteady solutions of the Euler equations evolve, and, even if these fixed points are unstable, their location in the function space may provide valuable clues concerning the structure of turbulent flow. The fact that there is apparently an incredibly rich variety of Euler flows corresponding to every possible streamline topology suggests that this idea may be worth pursuing.

Consider first the generic structure of an arbitrary solenoidal velocity field $\boldsymbol{U}(\boldsymbol{x})$ in $\mathscr{D}$, satisfying $\boldsymbol{n}\cdot\boldsymbol{U}=0$ on $\partial\mathscr{D}$. In some non-overlapping subdomains $\mathscr{D}_1$, $\mathscr{D}_2$, ..., $\mathscr{D}_n$ of $\mathscr{D}$, $\boldsymbol{U}(\boldsymbol{x})$ may be expected to be ergodic, and in the remainder of the domain the streamlines of $\boldsymbol{U}(\boldsymbol{x})$ will cover surfaces. Let $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ be an Euler flow that is topologically accessible from $\boldsymbol{U}(\boldsymbol{x})$ (this exists by the theorem of §8), and let $\boldsymbol{x}\rightarrow\boldsymbol{X}(\boldsymbol{x})$ be the corresponding mapping. Then, following the argument of §7,

$$\boldsymbol{\omega}^{\mathrm{E}}=\alpha_i\,\boldsymbol{u}^{\mathrm{E}}\quad \text{in } \hat{\mathscr{D}}_i \text{ (the image of } \mathscr{D}_i), \tag{9.1}$$

where $\alpha_i(\neq 0)$ is constant. Hence the relative helicity of the flow in $\hat{\mathscr{D}}_i$ is

$$\mathscr{H}_i=\frac{\int_{\hat{\mathscr{D}}_i}\boldsymbol{u}^{\mathrm{E}}\cdot\boldsymbol{\omega}^{\mathrm{E}}\,\mathrm{d}V}{\left\{\int_{\hat{\mathscr{D}}_i}\boldsymbol{u}^{\mathrm{E}2}\,\mathrm{d}V\int_{\hat{\mathscr{D}}_i}\boldsymbol{\omega}^{\mathrm{E}2}\,\mathrm{d}V\right\}^{\frac{1}{2}}}=\pm 1\quad\text{according as } \alpha_i\gtrless 0. \tag{9.2}$$

Thus in each ergodic 'blob' $\hat{\mathscr{D}}_i$, we have a flow of maximal helicity. These blobs may of course have horribly complicated boundaries, and they may be linked with each other in a complicated manner.

Let $\hat{\mathscr{D}}_{\mathrm{R}}$ denote that part of $\mathscr{D}$ that is not occupied by ergodic blobs; then in $\hat{\mathscr{D}}_{\mathrm{R}}$ the streamlines of $\boldsymbol{u}^{\mathrm{E}}$ lie on surfaces, and every point in $\hat{\mathscr{D}}_{\mathrm{R}}$ lies on such a streamsurface (figure 11). If any vortex sheets occur, these must be located on these streamsurfaces, as indicated in the figure.

This then, in schematic form, is the generic structure of an Euler flow. For obvious reasons, it may be expected to be in general unstable. In particular, any vortex sheets are subject to the Kelvin–Helmholtz instability and will wind up into characteristic double spirals. Nevertheless, the picture of figure 11 is suggestive in the context of turbulent flow. The blobs $\hat{\mathscr{D}}_i$ play the role of coherent structures, which do indeed in experimental contexts frequently exhibit strong helicity (positive or negative) (Tsinober & Levich 1983). The vortex sheets in $\hat{\mathscr{D}}_{\mathrm{R}}$ are the seat of viscous dissipation. The double-spiral instability to which these sheets are subject can in principle yield an inertial-range spectrum of Kolmogorov $(-\frac{5}{3})$ exponent (Moffatt 1984).

We may thus picture turbulence as a state at all times evolving in a neighbourhood of an Euler flow (which may be an Euler flow that is topologically accessible from the actual velocity field at a given instant), or possibly evolving through neighbourhoods of a succession of Euler flows. If there is any validity in this description, then high relative helicity $|\boldsymbol{u}\cdot\boldsymbol{\omega}|/(\boldsymbol{u}^2\boldsymbol{\omega}^2)^{\frac{1}{2}}$ should be correlated with low energy dissipation, and *vice versa*, a prediction that should be amenable to experimental and numerical testing.

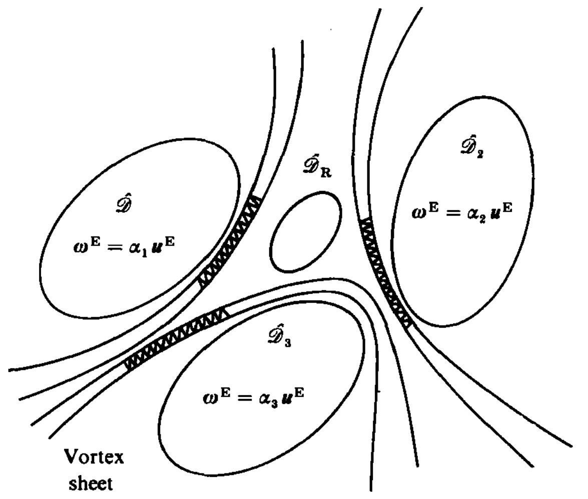


FIGURE 11. Schematic representation of generic structure of an Euler flow: 'coherent' helical structures (ergodic blobs) separated by the streamsurfaces on which vortex sheets may be located.

10. Conclusions and suggestions for further investigation

We have used the problem of magnetic relaxation as a vehicle for proof of the existence of magnetostatic equilibria, and hence of analogous Euler flows, of arbitrary topological structure. This arbitrary structure is defined by an 'initial' field $\boldsymbol{B}_0(\boldsymbol{x})$ which may be ergodic in certain regions and regular (i.e. having magnetic surfaces) in others. In the relaxation process the topological structure is conserved in the sense that $\boldsymbol{B}$-lines may not cross each other or reconnect; but magnetic surfaces may, and in general do, come together in the final stage of relaxation, forming tangential discontinuities (i.e. current sheets). The flow in this final stage has a squeeze-film character, in which the velocity falls to zero like an inverse power of time, and certain fluid elements experience unbounded total strain. The displacement mapping $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x})$ is then in the limit not a homeomorphism; it is this feature that in fact removes the severe structural constraints on magnetostatic equilibria (or Euler flows) implied by the theorems of Arnol'd (1965, 1966), which apply to analytic fields, and which do not therefore allow for the possible presence of tangential discontinuities.

The concept of 'topological accessibility', as opposed to 'topological equivalence', is natural in the present context, and is defined in §§2 and 3 in terms of a mapping $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x})$ that may be interpreted as the asymptotic displacement field associated with a smooth velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ $(0 < t < \infty)$ of finite total dissipation. The equilibrium field $\boldsymbol{B}^E(\boldsymbol{x})$ is then topologically accessible from $\boldsymbol{B}_0(\boldsymbol{x})$, and this means in particular that all knots and linkages in $\boldsymbol{B}_0(\boldsymbol{x})$ are mapped by $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x})$ into similar knots and linkages in $\boldsymbol{B}^E(\boldsymbol{x})$.

In the same sense there exists at least one Euler flow $\boldsymbol{u}^E(\boldsymbol{x})$ that is topologically accessible from an arbitrary solenoidal flow $\boldsymbol{U}(\boldsymbol{x})$ in $\mathscr{D}$. This Euler flow is characterized by 'blobs' of maximal helicity (positive or negative) in which the streamlines are ergodic, separated by regular regions that are dense with streamsurfaces, on some of which vortex sheets may be located. Such Euler flows are probably in general unstable, but they may nevertheless provide the basis for a structural model of turbulence, as suggested in §9.

Many problems arise from the present investigation and demand further study.

(i) The procedure of §3 provides an algorithm whereby magnetostatic equilibria that are topologically accessible from a given initial field $\boldsymbol{B}_0(\boldsymbol{x})$ may be (in principle) numerically determined. Of particular interest would be the formation of current sheets as $t \rightarrow \infty$. From a numerical point of view, the difficulty is to devise a procedure that works accurately at zero magnetic diffusivity. It may be necessary to retain a weak diffusive term $\eta \nabla^2 \boldsymbol{B}$ (with $\eta \ll \nu = \mu/\rho$) in (3.2) and to see whether a 'quasi-equilibrium' develops, with layers of high field gradient (presumably $O(\eta^{-\frac{1}{2}})$).

(ii) The squeeze-film behaviour in the final approach to magnetostatic equilibrium, in which two flux tubes approach each other, is of interest, and may be amenable to local analysis.

(iii) The question of uniqueness (or otherwise) of magnetostatic equilibria accessible through volume-preserving relaxation of a given field also deserves numerical investigation. As suggested in §3, variation of the viscosity parameter (or equivalently of the Reynolds number of the relaxation process) may lead to different equilibria. Moreover, different relaxation mechanisms (e.g. resistance proportional to velocity as in a porous medium) may be adopted to provide different routes towards equilibrium. Again we emphasize that the dissipative mechanism should be regarded more as a mathematical artifice than as an essential part of a real physical process.

(iv) The magnetic relaxation problem may equally be formulated in a *compressible* medium (having both shear and bulk viscosity). If, for example, pressure and density are related by $\mathrm{d}p/\mathrm{d}\rho = k$, where k is constant, then we arrive at a *family* of equilibria $\boldsymbol{B}_k^{\mathrm{E}}(\boldsymbol{x})$ $(0 < k < \infty)$, the limit $k \rightarrow \infty$ corresponding to the incompressible case, and the limit $k \rightarrow 0$ corresponding to a 'pressureless plasma'. The analogy with Euler flows of an incompressible inviscid fluid (which is evidently still at our disposal) then implies the existence of a corresponding *family* $\boldsymbol{u}_k^{\mathrm{E}}(\boldsymbol{x})$ of Euler flows topologically accessible (via mappings that are no longer volume-preserving) from a given flow $\boldsymbol{U}(\boldsymbol{x})$. This opens up a wealth of possibilities which will be the subject of a separate paper. The limit $k \rightarrow 0$ (for which the equilibrium field is a Beltrami field) has peculiar properties which will require special consideration.

(v) The restriction $\boldsymbol{B}_0 . \boldsymbol{n} = 0$ on $\partial \mathscr{D}$ that has been adopted in this paper may also be relaxed. If we adopt an initial condition of the form $\boldsymbol{B}_0 . \boldsymbol{n} = f(\boldsymbol{x})$ on $\partial \mathscr{D}$, where $\int_{\partial \mathscr{D}} f(\boldsymbol{x})\, \mathrm{d}S = 0$, then during the relaxation process, $\boldsymbol{B} . \boldsymbol{n} = f(\boldsymbol{x})$ for all $t > 0$, and the equilibrium (relaxed) field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ satisfies this boundary condition also. This formulation of the problem (particularly if carried out in a compressible medium) has an immediate bearing on the problem of the adjustment of a magnetic field in a given domain in response to movement of the 'footpoints' of the $\boldsymbol{B}$-lines where these intersect the boundary. This is a large and important problem in the context of solar-flare theory (see e.g. Sweet 1969) which also deserves separate treatment.

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